Contourlet domain image modeling by using the alpha-stable family of distributions

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Abstract—**It is known that the contourlet coefficients of images have non-Gaussian property and heavy tails. In view of this, an appropriate distribution to model the statistics of the contourlet coefficients would be the one having large peaks, and tails heavier than that of a Gaussian PDF, i.e., a heavy-tailed PDF. This paper proposes a new image modeling in the contourlet domain, where the magnitudes of the coefficients are modeled by a symmetric alpha-stable distribution which is best suited for modeling transform coefficients with a high non-Gaussian property and heavy tails. It is shown that the alpha-stable family of distributions provides a more accurate model to the contourlet subband coefficients than the formerly used distributions, namely, the generalized Gaussian and Laplacian distributions, both in terms of the subjective measure of the Kolmogorov-Smirnov distance and the objective measure of comparing the log-scale histograms.**

Index Terms—Contourlet transform, alpha-stable distribution, statistical image modeling, amplitude probability density function.

I. INTRODUCTION

There are many works on image modeling in the transform domain by using the statistics of the subband coefficients of the wavelet transform [1], [2]. In recent years, the contourlet transform has received much attention and has been regarded as an alternative to other multi-scale and multi-resolution transforms, like wavelet transform, in many image processing applications such as image denoising, image watermarking, text retrieval and feature extraction [3]-[6]. This is mainly because of its appealing characteristics in capturing smooth contours and geometric structures in images [3], [4]. Therefore, researchers have studied the statistical properties of the contourlet subband coefficients and have found that the contourlet subband coefficients of images have significantly non-Gaussian and heavy-tailed properties that are best described by heavy tailed distributions [3]. The contourlet coefficients within a subband may be assumed to be independent and identically distributed. With this assumption, researchers have developed a marginal model for the contourlet subband coefficients using the generalized Gaussian density (GGD) [3], [5], [7].

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Although, the GGD is a suitable distribution for the peaky and heavy-tailed non-Gaussian statistic of typical image contourlet decomposition, it may not be the best choice for modeling the contourlet coefficients. In view of this, we develop a new model reflecting these properties, which is useful in current contourlet applications including watermarking and denoising. The alpha-stable family of distributions is proposed to characterize the contourlet coefficients. It has been shown that this model is suitable for describing signals that have highly non-Gaussian distributions with heavy tails [8]. The characteristic exponent, α of the alpha-stable distribution exhibits various degrees of non-Gaussian characteristics in different directional subbands of the contourlet transform. We observe that distributions with heavy algebraic tails, such as the alpha-stable family, are more appropriate distributions for modeling the contourlet coefficients of natural images than families with exponential tails such as the generalized Gaussian. The amplitude probability density function is also used to verify the accuracy of our proposed model. This function can be obtained in a straightforward manner by counting the contourlet coefficients and by estimating the distribution parameters from the transformed coefficients. It is shown that the alpha-stable model fits more accurately the empirical data, as compared to formerly used distributions, namely, the generalized Gaussian and the Laplacian distributions, both in terms of the subjective measure of the Kolmogorov-Smirnov distance and objective measure of comparing the log-scale histograms.

II. CONTOURLET IMAGE MODELING

A. *Statistical properties of contourlet coefficients*

It is known that the contourlet coefficients have high non-Gaussian property [1], [3]. This point can be justified by studying the marginal statistics of the contourlet coefficients of natural images. Suppose an image is decomposed into J scales and D direction subbands by the contourlet transform and subbands are denoted by S_{jd}, where $j=1,...,J$ and $d=1,...,D$. In Fig. 1, histograms of the contourlet coefficients of two finest scales with eight directions in each scale for the *Barbara* image are plotted. Compared to a Gaussian, these densities are more sharply peaked and with more extensive tails. To quantify this, we specify the sample kurtosis [9] (the fourth moment divided by the squared second moment) above each histogram.

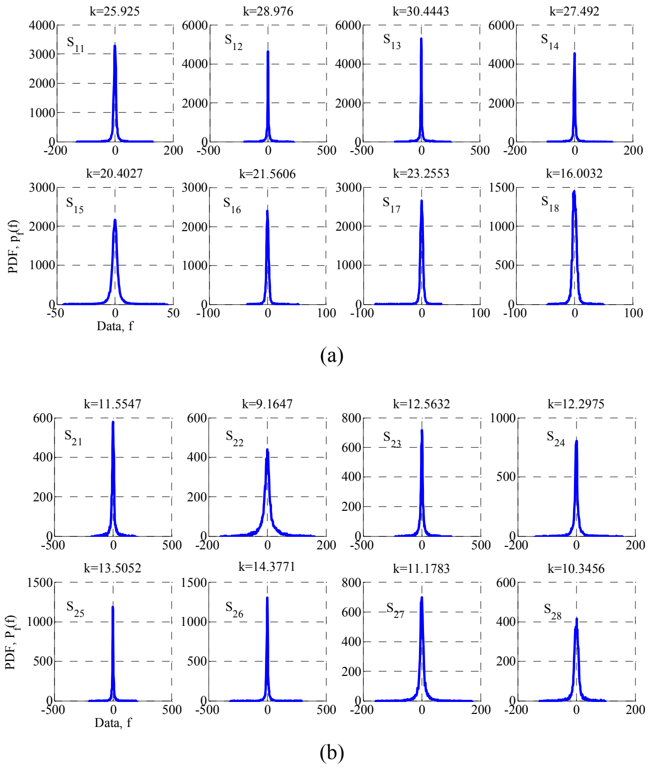

Fig. 1. Histograms of the contourlet subband coefficients for two finest scales of the *Barbara* image. The kurtosis values k shows the degree of non-Gaussianity for coefficients of each subband; S_{jd} denotes a subband in j th scale and d th direction… (a) the finest scale. (b) the second finest scale.

Similar results are also observed for other scales and other test images which demonstrate the non-Gaussianity of the contourlet coefficients of natural images i.e., having large peaks and tails heavier than that of the Gaussian PDF. Fig. 2 shows average kurtosis of the contourlet subband coefficients of 96 images, obtained from [10], for various values of the number of subbands as a function of the scale number. It is seen from this figure that average kurtosis value is close to, yet greater than, the kurtosis value of the Gaussian distribution, i.e., 3, when the number of scales is increased. It reveals that in higher scales of a multi-scale representation like the contourlet transform, the distribution is much closer to the normal. Therefore, one needs to choose an appropriate number of scales for modeling of the transformed domain coefficients by a non-Gaussian distribution.

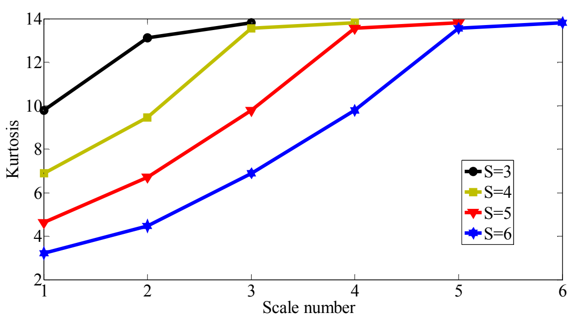

Fig. 2. Average kurtosis values of contourlet subband coefficients for various values of the number of subbands S .

B. *Alpha-stable family of distributions*

In this section, we briefly describe the alpha-stable statistical model that will be used to characterize the contourlet image coefficients. This model is suitable for describing signals with non-Gaussian statistics and heavy tails [8]. A random variable $X \sim S_\alpha(\gamma,\beta,\delta)$ is distributed with alpha-stable distribution can be best described by its characteristic function

$$\varphi(\omega) = \exp\left\{ j\delta\omega - \gamma|\omega|^{\alpha}[1 + j\beta sign(\omega)\varpi(\omega,\alpha)]\right\} \tag{1a}$$

with

$$\varpi(\omega,\alpha) = \begin{cases} \tan\dfrac{\alpha\pi}{2} & if \alpha \neq 1 \\ \dfrac{2}{\pi}\log|\omega| & if \alpha = 1 \end{cases} \tag{1b}$$

where α is a characteristic exponent, $0 < \alpha \leq 2$, $\beta \in [-1,1]$ is a skewness parameter, $\delta \in \Re$ is a location parameter, and $\gamma > 0$ is a dispersion parameter. For values of $1 < \alpha \leq 2$, δ corresponds to the mean of the distribution, while for $0 < \alpha \leq 1$, δ corresponds to its median. A stable distribution is called standard if $\delta = 0$ and $\gamma = 1$. Another important particular case of stable distributions is obtained for $\beta = 0$. In this case, the distribution is symmetric about the center δ. Symmetric stable distributions with characteristic exponent α are called symmetric alpha-stable or $S\alpha S$. The characteristic exponent α is the most important parameter which determines the shape of the distribution. The smaller the value of α, the heavier the tail of the distribution. This implies that random variables following the $S\alpha S$ distribution with small characteristic exponents are highly impulsive. By using power series expansions, the $S\alpha S$ density function for a random variable is given by

$$f_{\alpha,\gamma}(x) = \begin{cases} \dfrac{1}{\pi\gamma^{1/\alpha}}\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\Gamma(\alpha k+1)\sin\left(\frac{k\alpha\pi}{2}\right)\left(\frac{|x|}{\gamma^{1/\alpha}}\right)^{-\alpha k-1} & 0<\alpha<1 \\ \dfrac{\gamma}{\pi(x^2+\gamma^2)}, & \alpha=1 \\ \dfrac{1}{\pi\alpha\gamma^{1/\alpha}}\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(2k)!}\Gamma\left(\frac{2k+1}{\alpha}\right)\left(\frac{x}{\gamma^{1/\alpha}}\right)^{2k} & 1<\alpha<2 \\ \dfrac{1}{2\sqrt{\pi\gamma}}\exp\left(-\dfrac{x^2}{4\gamma}\right), & \alpha=2 \end{cases} \tag{2}$$

It is to be noted that there is no closed-form expression for the $S\alpha S$ distribution except when α takes the values 1 or 2, thus defining two special cases of $S\alpha S$ distributions, namely, the Cauchy $(\alpha = 1)$ and the Gaussian $(\alpha = 2)$ distributions.

C. Alpha-stable modeling of contourlet coefficients

The symmetric alpha-stable family of distributions has attracted attention in the modeling of heavy-tailed data such as the transform-domain image coefficients. In order to model the contourlet subband coefficients of an image, we propose the use of $S\alpha S$ distribution. To this end, we estimate the values of the characteristic exponent α, for the various contourlet subbands for a given test image. Here, the image is decomposed into three pyramidal levels ($J = 3$), with eight, eight and four directions, respectively. In Table I, the results obtained by estimating α by using the maximum likelihood approach [11] for three of the test images, *Peppers*, *Barbara* and *Baboon*, are summarized. It may be mentioned that there are several other estimators that can be used to estimate the parameters of an alpha-stable distribution such as the characteristic function method and the quantile estimation method [8]. It can be seen from Table I that the values of α vary from 0.8 to 1.6 indicating the heavy-tailed property of the contourlet coefficients. Thus, the distribution of the contourlet coefficients of an image can be described by a $S\alpha S$ PDF. We then investigate as to how accurately the $S\alpha S$ distribution fits the distribution of the contourlet coefficients. For this purpose, we examine the histograms of the actual data as well as the $S\alpha S$, the GG and the Laplacian density functions for a number of test images. The modeling performance of the contourlet coefficients for one of the images, *Barbara*, is shown in Fig. 3. It is evident from this figure that the $S\alpha S$ distribution fits the empirical data much more accurately than the GG and Laplacian distributions do.

TABLE I
THE VALUES OF THE CHARACTERISTIC EXPONENT α, FOR THE CONTOURLET SUBBAND COEFFICIENTS OF VARIOUS TEST IMAGES. DEVIATION FROM $\alpha = 2$ REVEALS THE DEGREE OF NON-GAUSSIANITY.

Direction	Scale	Peppers	Barbara	Baboon
1	I	1.516	1.436	1.577
2		1.506	1.358	1.514
3		1.571	1.337	1.562
4		1.541	1.308	1.538
1	II	1.330	1.376	1.471
2		1.253	1.335	1.475
3		1.260	1.392	1.484
4		1.403	1.334	1.406
5		1.407	1.106	1.342
6		1.294	1.261	1.256
7		1.211	1.203	1.225
8		1.313	1.423	1.335
1	III	1.432	0.966	1.341
2		1.336	1.024	1.467
3		1.196	0.848	1.312
4		1.216	1.105	1.523
5		1.225	1.375	1.412
6		1.217	1.264	1.332
7		1.115	1.209	1.314
8		1.388	1.591	1.425

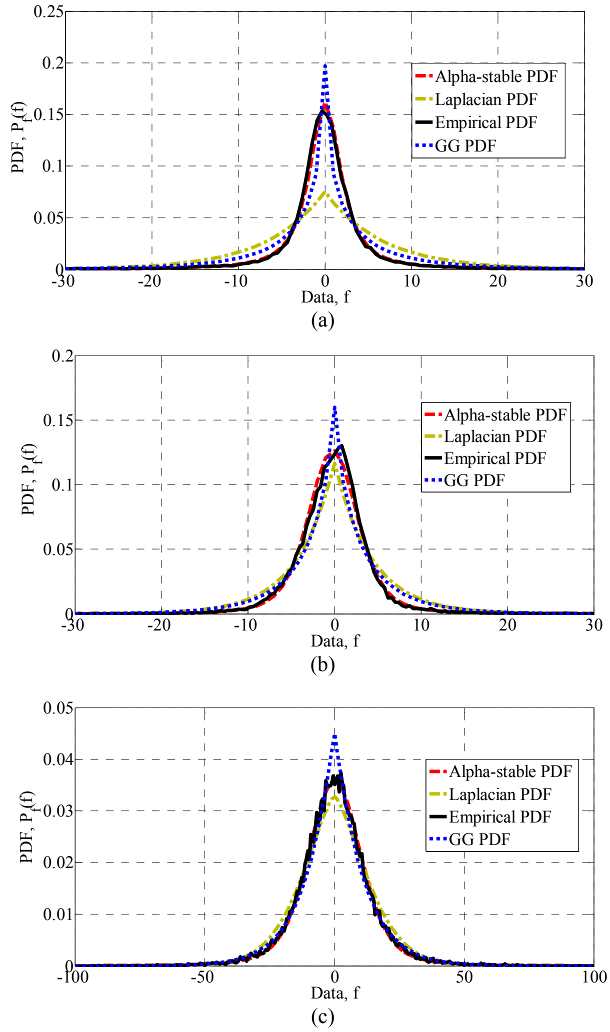

Fig. 3. PDFs of empirical data as well as the alpha-stable, GG and Laplacian distributions for a) *Barbara* and b) *Peppers* c) *Baboon* images.

The amplitude probability density (APD) function, given by $P(|X| > x)$, is another common statistical representation of heavy-tailed signals. The APD can be used to compare the closeness of the $S\alpha S$ distribution to the empirical data. It can be empirically calculated by counting the data, X for which $|X| > x$. It can also be evaluated theoretically from a given density function by estimating its parameters from the transformed coefficients. It is known that the alpha-stable density function has a polynomial tail $P(X > x) \sim c_\alpha x^{-\alpha} \gamma^\alpha$, $x \to \infty$, where X is a non-Gaussian $S\alpha S$ random variable and $c_\alpha = \sin(\frac{\pi\alpha}{2})\frac{\Gamma(\alpha)}{\pi}$ [12]. We now examine the APD curves of the actual data as well as the $S\alpha S$, GG and Laplacian distributions for a number of different images. In Fig. 4 the APD curves for one of the images, *Barbara,* is depicted. It is seen in this figure that the $S\alpha S$ PDF provides a better fit for the distribution of the contourlet coefficients than that provided by the GG and the Laplacian distributions for both the mode and the tail of the actual data.

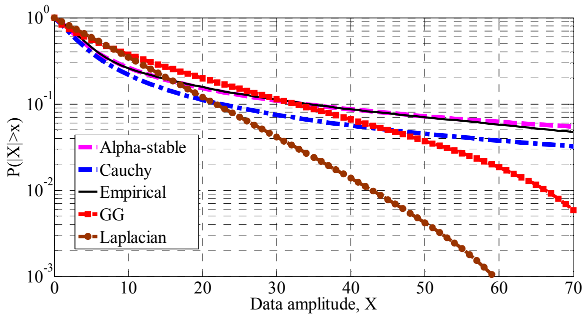

Fig. 4. PDFs of empirical data as well as the alpha-stable, GG and Laplacian distributions for *Barbara* images.

Moreover, to quantify the performance of the PDFs, we use, the Kolmogorov-Smirnov distance (KSD) metric given by

$$\max_{f}\left|\int[P_f(f)-\hat{P}_f(f)]df\right| \tag{4}$$

in which, $P_f(f)$ denotes the PDF of the random variable and $\hat{P}_f(f)$ represents the PDF of the empirical data. Table II shows the results concerning the metric KSD for the alpha-stable and GG PDFs of the image contourlet coefficients in three finest scales. The values of the metric are obtained for three of the test images *Peppers*, *Barbara* and *Baboon*, each of size 512×512. The values of the KSD in Table II indicate that the $S\alpha S$ distribution provides a better fit as compared to the GG distribution.

TABLE II
VALUES OF THE KOLMOGOROV-SMIRNOV DISTANCE FOR VARIOUS CONTOURLET SUBBAND COEFFICIENTS.

Direction	Scale	Peppers		Barbara		Baboon	
		GG	SαS	GG	SαS	GG	SαS
1	**I**	0.0387	**0.0350**	0.0398	**0.0397**	0.0361	**0.0343**
2		0.0567	**0.0442**	0.0449	**0.0380**	0.0415	**0.0258**
3		0.0531	**0.0315**	0.0615	**0.0407**	0.0416	**0.0374**
4		0.0588	**0.0423**	0.0397	**0.0320**	0.0654	**0.0588**
1	**II**	0.0996	**0.0436**	0.0393	**0.0341**	0.0447	**0.0282**
2		0.0841	**0.0313**	0.0395	**0.0371**	0.0363	**0.0249**
3		0.0812	**0.0375**	0.0372	**0.0367**	0.0353	**0.0236**
4		0.1225	**0.0560**	0.0846	**0.0452**	0.0376	**0.0214**
5		0.1423	**0.0481**	0.0691	**0.0505**	0.0286	**0.0283**
6		0.0869	**0.0406**	0.1058	**0.0389**	0.0376	**0.0365**
7		0.0951	**0.0450**	0.0671	**0.0419**	0.0480	**0.0469**
8		0.1117	**0.0284**	0.0683	**0.0331**	0.0443	**0.0309**
1	**III**	0.1025	**0.0482**	0.1290	**0.0508**	0.0332	**0.0164**
2		0.1684	**0.0578**	0.0568	**0.0468**	0.0517	**0.0222**
3		0.1809	**0.0656**	0.1325	**0.0681**	0.0476	**0.0228**
4		0.1427	**0.0709**	0.1266	**0.0836**	0.0347	**0.0243**
5		0.1170	**0.0437**	0.1764	**0.0920**	0.0365	**0.0170**
6		0.1615	**0.0637**	0.1850	**0.0954**	0.0355	**0.0229**
7		0.1412	**0.0597**	0.1745	**0.0870**	0.0465	**0.0250**
8		0.0960	**0.439**	0.1346	**0.0556**	0.0287	**0.0147**

Motivated by our modeling results, in future, we intend to design new watermarking and denoising schemes in the contourlet domain. The proposed alpha-stable model is intended to be used in developing an additive or a multiplicative watermarking scheme that would improve the performance of watermark detection. Also, this model would be considered as a prior for developing a maximum *a posterior* estimator in an image denoising scheme. It is expected that a better statistical model would result in improved watermark detection and denoising algorithms.

III. CONCLUSION

In this work, the statistical properties of the contourlet coefficients of images have been investigated and the empirical PDFs of the contourlet coefficients of images have been shown to best fit the alpha-stable distribution. The performance of the proposed model has been studied in detail by conducting several experiments, and comparing the results with that of the formerly used distributions for the contourlet coefficients, namely, the generalized Gaussian and the Laplacian PDFs. Simulation results have shown that the alpha-stable distribution can model the contourlet subband coefficients more accurately both subjectively in terms of the Kolmogorov-Smirnov distance and objectively by plotting the log-scale histograms. Moreover, comparing the amplitude probability density functions of the various distributions have shown that the alpha-stable distribution provides a better fit for the distribution of the contourlet coefficients than that provided by for both the mode and the tail of the actual data.

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